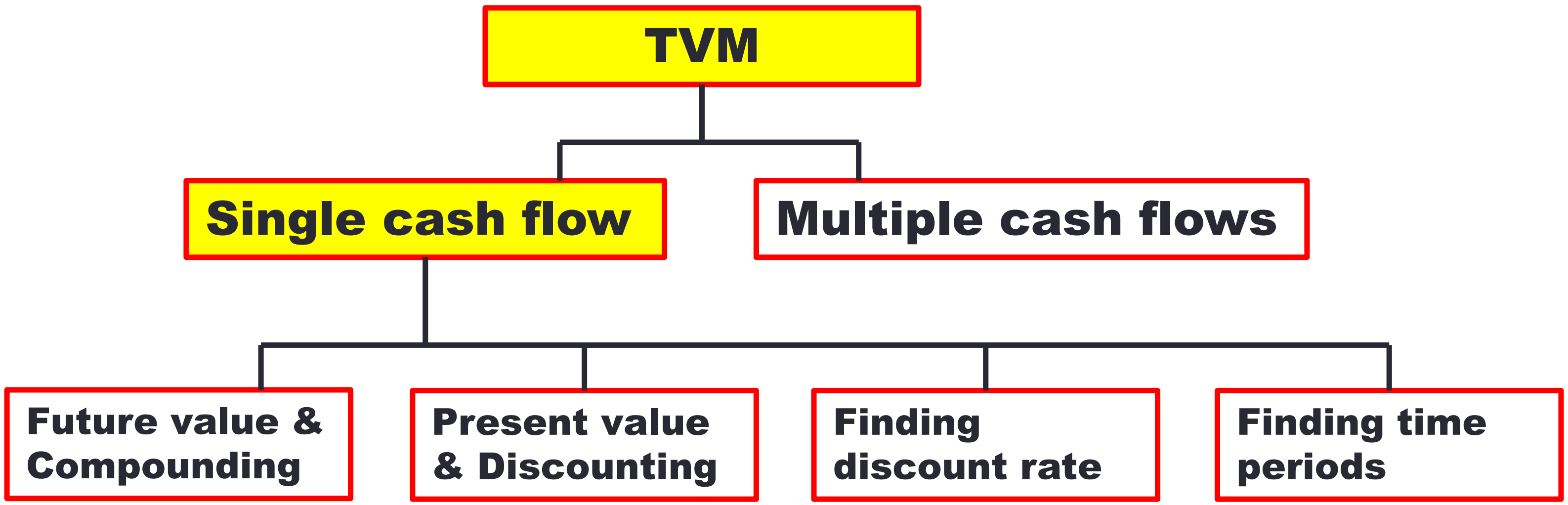
TIME VALUE OF MONEY

Time value of money I



Guess the answer

If the interest rate has increased from 0.1% to 3%, how do investors change their required rate of return for their investment?

- A. Increase B. Decrease C. No change

Suppose Investor A invests 100 dollars for one year at 10% per year.

Investor B invests 100 dollars at 5% for two years. Who reaps more value from her investment?

- A. Investor A B. Investor B C. The same

The Concept of TV of Money

a dollar today has more value than a dollar tomorrow

- 1 dollar has more purchase power in 1900 than in 2020

Present value

Future value

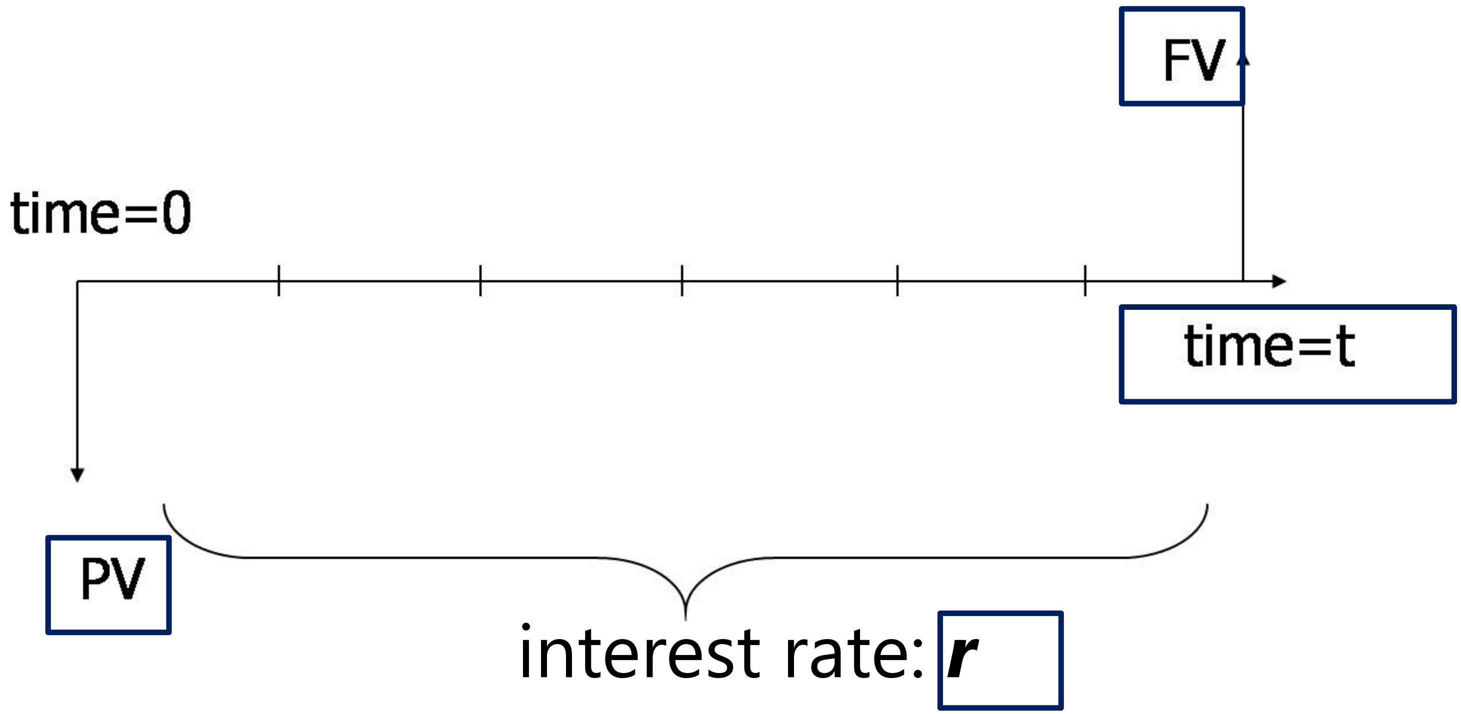
Time Value of Money

- Timing of the payment matters!
- It's better to receive a sum of money **today** than to receive the same tomorrow
  - Because it can be invested and earn interest
  - Given an interest rate, we can compute the present value (today's money) of a sum to be received in the future.

Watch
https://www.youtube.com/watch?v=1CO5hf2ns18&ab_channel=WallStreetSurvivor

Think:
Would Natalie take 1025,000 award one year after if the interest rate is 2%.

Timeline of Investment



Present value (*PV*)

the current value of future cash flows discounted at the appropriate discount rate
value at $t = 0$ on a timeline

Future value (*FV*)

the amount an investment is worth after one or more periods
"later" money on a timeline

Interest rate (r)

Interest rate $(\boldsymbol{r})$ is the rate of return required by investors.

- A.K.A
  - discount rate
  - **cost of capital**
  - **opportunity cost of capital (return on alternatives)**
  - required rate of return

Understanding Financial Activities

- Before we start to talk about the examples of financial problems, please finish the pre-course quiz ("Short quiz – glossary").

Financial Activities of Cash Flows

Activities leading to cash outflows (money flowing out of your pocket)

- Investments (buying shares of Starbucks, buying US government bonds, lending money)
- Savings (depositing money in a bank saving account)
- Contributions to pension funds (saving money in a pension fund), trust funds, or college funds

Activities leading to cash inflows (money flowing into your pocket)

- Receiving dividends or interest payments
- Selling/issuing the shares
- Selling/issuing bonds (corporate borrowing), obtaining a loan
- Withdrawing money
- Receiving pension payments or fund payments

Future value – one period example

- Suppose you invest $100 for one year at 10% per year
- What is the future value in one year (how much can you yield in one year)?

  - Interest

  $= 100 \times 0.10 \qquad = 10$

  - Value in one year

  $=$ principal + interest $\qquad = 100+10=110$

  - Or (Future value) $\qquad = 100 \times (1+0.10)=110$

Future value – two period example

Suppose you invest \$100 for two years at 10% per year. What is the future value in year 2?

It depends what you do with the interest (100 x 10%=10) at the end of period 1

- Withdraw \$10 interest at the end of period 1 and leave \$100 (your balance is always equal to 100)

$$FV = 100 + 100 \times 0.1 + 100 \times 0.1 = 100 \times (1 + 0.1 \times 2) = 120$$

- $FV = PV + PV \times r + PV \times r + \ldots + PV \times r = PV \times (1 + r \times t)$

Future value – two period example

Suppose you invest \$100 for two years at 10% per year. What is the future value in year 2?

It depends what you do with the interest (100 x 10%=10) at the end of period 1

- Leave the \$10 interest and \$100
  - First period: $100 + 100 \times 0.1 = 110$ or $100 \times (1+10\%)^{1} = 110$
  - Second period: $110 + 110 \times 0.1 = 121$ or $100 \times (1+10\%)^{2} = 121$
- $$FV = PV \times (1 + r)^{t}$$
- Where is the extra \$1 from?
- From interest on the 10\$ interest

Compounding

- Simple interest
  - interest earned only on the original principal
  - Interest is not reinvested
- Compound interest
  - interest earned on principal and on interest received
  - "interest on interest" – interest earned on **reinvestment of previous interest** payments

- Consider the previous example $(t = 2, r = 10\%)$
  - FV with simple interest = 100 × (1+10%×2) = 120
  - FV with compound interest =100×(1+10%)^2 = 121
  - the extra $1 comes from the interest of 0.10×10=1 earned on the first interest payment

Future value

- FV with **compound** interest

$$FV = PV \times (1 + r) \times (1 + r) \times \cdots \times (1 + r)$$
$$FV = PV \times (1 + r)^t$$

- FV with simple interest

$$FV = PV + PV \times r + PV \times r + \cdots + PV \times r$$
$$FV = PV \times (1 + (r \times t))$$

Future value (FV) Formula

The amount of money an investment will grow to over some period of time at a given interest rate:

$$FV = PV \times (1 + r)^t$$

FV - future value

PV - present value

r - period interest rate, expressed as a decimal

t - number of periods ***(n)***

Future value – example

Suppose you had deposited $5 for you at 6% interest 200 years ago.

How much would the investment be worth today by compounding interest?

$$= PV \times (1 + r)^t = 5 \times (1 + 0.06)^{200} = 575{,}629.52$$

How much can you get if the investment only earns simple interest?

$$= PV \times (1 + (r \times t)) = 5 + 5 \times (0.06 \times 200) = \$65$$

- The difference is amazing!
- The effect of compounding is small for a small number of periods, but increases as the number of periods increases.

Future value – important relationships

$$FV = PV \times (1 + r)^t$$

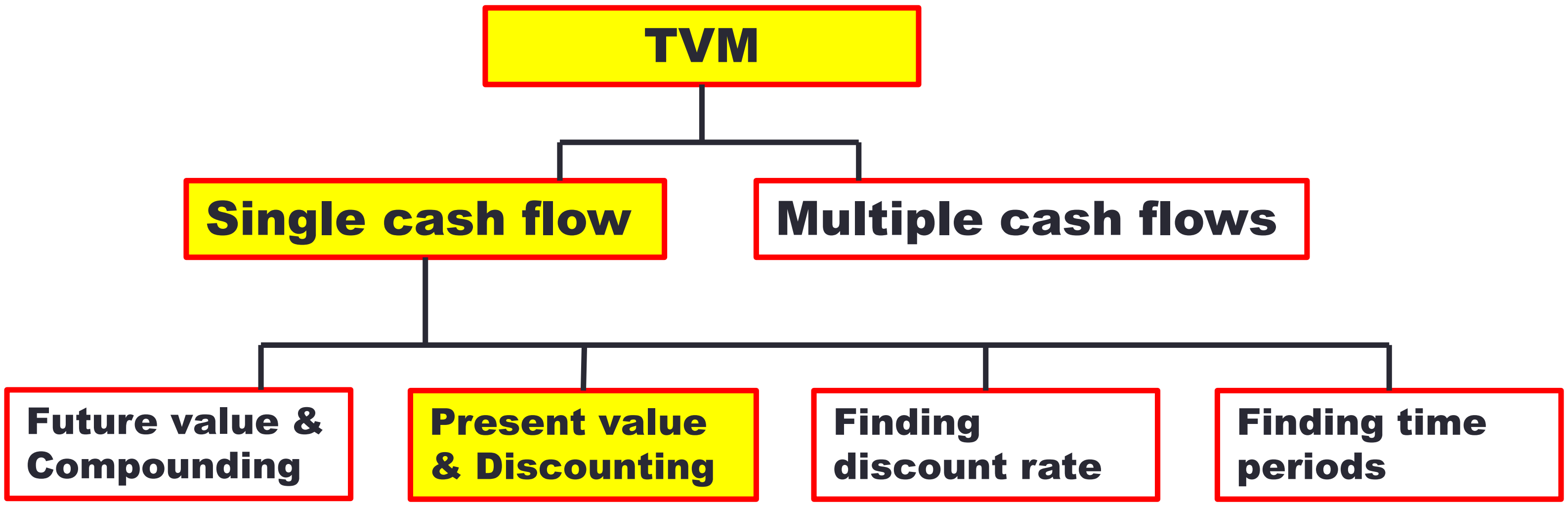
- Other things equal:
- The longer the time period, the higher the future value
  - If you invest \$500 in 5 years at an interest rate of 10%?
  - \$805.26
  - If you invest \$500 in 10 years at an interest rate of 10%?
  - \$1,296.87
- The higher the interest rate, the larger the future value
  - Value of investing \$500 in 5 years if the interest rate is 10% and 15%?
  - \$805.26 and \$1,005.67

Go back to pre-course quiz

If the interest rate has increased from 0.1% to 3%, how do investors change their required rate of return for their investment?

Suppose Investor A invests $100 for one year at 10% per year. investor B invests $100 at 5% for two years. Who reaps more value from her investment?

Time value of money I



Based on $FV = PV \times (1 + r)^t$, rearrange and solve PV, r, t

Present value *(PV)*

- Present value refers the current value of an amount to be received in the future
- Value at $t = 0$ on a time line, and it capture today's purchase power.
- **Discounting** means finding the present value of one or more future amounts

Present value *(PV)*

How much do I have to invest today to have some amount in the future?

$$FV = PV \times (1 + r)^t$$

- Rearrange to solve for $\boldsymbol{PV = \frac{FV}{(1+r)^t}}$

- When we talk about the **"value"** of something, we are talking about the present value unless we specifically indicate that we want the future value
- Example: What is the value of my pension pot that pays me a lump-sum of 1 million in 50 years if the interest rate is 10%?
-

Present value – example 1

You want to begin saving for your daughter's college education and you estimate that she will need 150,000 in 17 years′ time. If you feel confident that you can earn 8% per year, how much do you need to invest today?

$$PV = \frac{FV}{(1+r)^t}$$

$$\frac{150{,}000}{(1+0.08)^{17}} = 40{,}540.34$$

Present value – example 2

Your parents set up a trust fund for you 10 years ago that is now worth $19,671.51. If the fund earned 7% per year, how much did your parents invest 10 years ago? (have a think about it)

Present value – important relationships

$$PV = \frac{FV}{(1 + r)^t}$$

- For a given interest rate, the longer the time period, the lower the present value
  - for a given r, as t increases, PV decreases
- For a given time period, the higher the interest rate, the smaller the present value
  - for a given t, as r increases, PV decreases

Base on the relationship..

Which option would you prefer (giving you the highest PV)

- 1000 dollars in 10 years
- 1000 dollars in 3 years

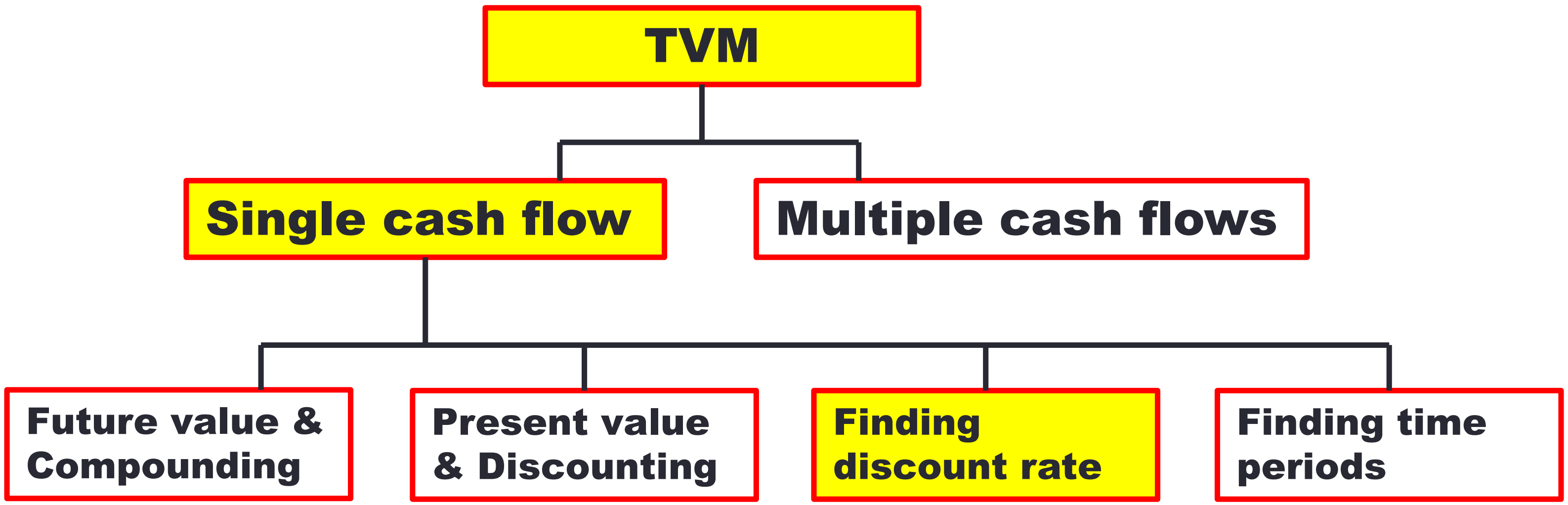
Which option would you prefer (giving you the highest PV)

- Being an investor in US being promised to receive 1000 dollars in 10 years, where the local interest rate is 5%
- Being an investor in Japan being promised to receive 1000 dollars in 10 years, where the local interest rate is 0.5%

PV or FV

- You want to sell your car.
- You are offered $10,000 for your car today and
- You also received another offer of $11,424 but the second payment is due in one year’s time.
- Interest rates are **12%** and both offers come from financially reliable sources. Which one do you accept? Think about it for 5 minutes.

Time value of money I



Discount rate *(r)*

- We often want to know what the implied interest rate is in an investment
- Recall that the basic equation: $PV = \frac{FV}{(1+r)^t}$
- Rearrange the basic *FV* equation and solve for *r* :

$$FV = PV \times (1+r)^t \quad \rightarrow \quad \boldsymbol{r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1}$$

Discount rate - example

You are looking at an investment that will pay 1,200 in 5 years if you invest 1,000 today. What is the implied rate of interest?

Time value of money I

- **TVM**
  - **Single cash flow**
    - **Future value & Compounding**
    - **Present value & Discounting**
    - **Finding discount rate**
    - **Finding time periods**
  - **Multiple cash flows**

Number of periods (*t* or *n*)

- Start with basic equation and solve for *t* (remember your logs)

$$FV = PV \times (1+r)^t$$

$$\frac{FV}{PV} = (1+r)^t$$

$$\ln\left(\frac{FV}{PV}\right) = t \times \ln(1+r)$$

$$t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$$

Number of periods - example

You want to purchase a new car and you are about to pay €20,000.

If you can invest at 10% per year and you currently have €15,000, how long will it be before you have enough money to pay for the car?

Try to solve it.

Number of periods - example

You want to purchase a new car and you are willing to pay €20,000. If you can invest at 10% per year and you currently have €15,000, how long will it be before you have enough money to pay cash for the car?

$$t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$$

$$\ln\left(\frac{FV}{PV}\right) = \ln\left(\frac{20{,}000}{15{,}000}\right) = \ln(1.333) = 0.2877$$

$$\ln(1+r) = \ln(1+0.1) = 0.0953$$

$$t = \frac{0.2877}{0.0953} = 3.02\ years$$

The rule of 72

A quick way of checking how long it will take you to **double** your money given an interest rate and compound interest.

How long will it take to double $1,000 at an interest rate of 8%? Consider the answers from the PV formula and the rule of 72.

$$\text{Precise answer} = \frac{\ln(2000/1000)}{\ln(1.08)} = 9.006468$$

The rule of 72: 72/8 = 9 years

Group Exercise

Q1. Suppose you are offered an investment that will allow you to double your money in 6 years. You have 10,000 to invest. What is the implied rate of interest?

Q2. Suppose you have a 1-year old son and you want to provide $75,000 in 17 years towards his college education. You currently have $5,000 to invest. What interest rate must you earn to have the $75,000 when you need it?

Q3. Suppose you want to buy a new house. You currently have 15,000 and you figure you need to have a 10% down payment plus closing costs. *Closing costs are 15% of total borrowing.* If the type of house you want costs about 150,000 and you can earn 7.5% per year, how long will it be before you have enough money for the down payment and closing costs?

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Annuity Excel Spreadsheet Functions

Using Microsoft Excel – the sign convention matters!!!

- For example: You invest 1000 dollars today and will receive 1200 dollars in 5 years, to calculate the rate of investment:
  - NPER = 5 (number of periods, you receive money in 5 years)
  - PV = – 1000 (you invest 1000 today, cash outflow)
  - FV = 1200 (you receive or withdraw 1200 in 5 years, cash inflow)
  - PMT = 0 (at this moment, we have zero payment per periods…)
  - Use RATE function: = rate(nper, pmt, pv, fv) = rate(5,0,–1000,1200) = 3.714%
- to calculate the number of periods(t) knowing the discount rate is 3.714%
  - =NPER(0.03713,0,–1000,1200)
- to calculate the PV knowing the discount rate is 3.714%
  - =–PV(0.03714,5,0,1200)
- to calculate the FV knowing the discount rate is 3.714%
  - = FV(0.03714,5,0, –1000)

Q4. Excel Exercise

Download Excel Exercise from Google Excel (https://docs.google.com/spreadsheets/d/1RoEkQ6AQlq0J5hnGjfLgGcUP0Neu0OT30Sr7E_LsuNA/edit?usp=sharing)

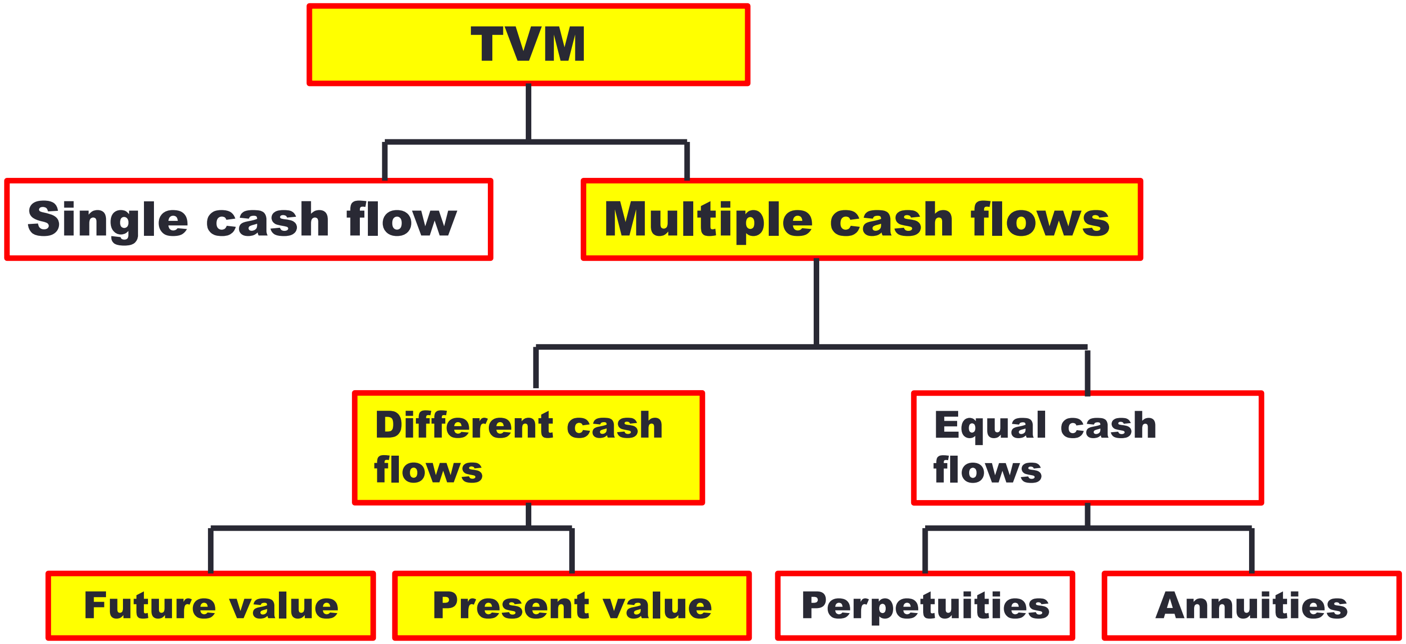
Fill in the blank with formula and Excel functions. Recommend you to write down the formula, for example: $\underline{81550 \times (1 + 12\%)^{17} \quad FV(0.12, 17, 0, -81550)}$

Present Value	Years	Interest Rate	Future Value
81550	17	12%	
	14	22%	886073
1000	5		2000
221	4		307
250		4%	1105
1000		5%	2000
12000	10	10%	
	10	12%	1000

The formula sheet of Excel will be Provided in the exam.

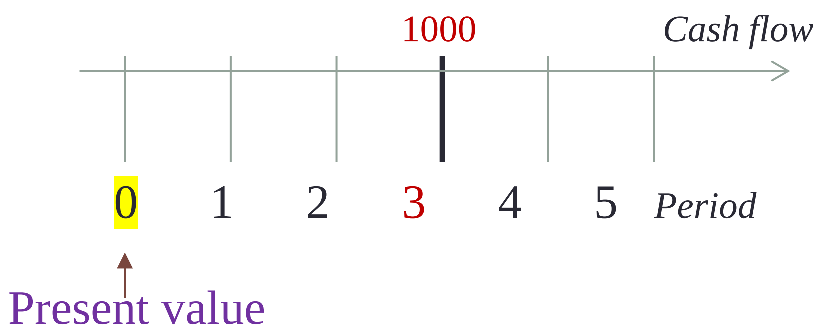
Formula Sheet of Excel:
Payment per period: PMT (rate,nper,pv,[fv],[type])
Present value: PV (rate, nper, pmt,[fv],[type])
Future value: FV (rate, nper, [pmt], [pv], [type])
Number of payment: NPER (rate, pmt, pv, [fv], [type])
Net Present Value: NPV (rate, value1, value 2,…)
Internal rate of return: IRR(value 1, value 2,.)

Time value of money

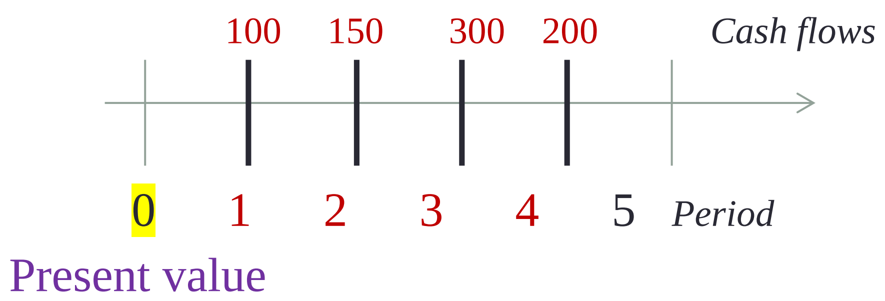


Multiple cash flows scenario

- Single cash flow



- Multiple cash flows



Multiple cash flows – Future value

1) We plan to save \$1000 today and at the end of the next two years. At a fixed 10% rate, how much will we have in the bank three years from today?

- Answer: 3641\$

2) Based on 1), If we plan to invest a lump-sum amount today (and make no further payments) and receive the same amount after 3 years (which is 3641\$) , what amount should I invest?

0 1 2 3

1000 1000 1000

FV = $1000x(1 + r)^3$=1331

$1000x(1+r)^2$ =1210

1000x(1+r)=1100

\$3641

PV= FV / $(1 + r)^n$=

3641/ $(1 + .10)^3$ =**\$2735.54**

Multiple cash flows – Future value

You think you will be able to deposit 4,000 at the end of each of the next three years in a bank account paying 8% interest

You currently have 7,000 in the account

How much will you have in 3 years?

$$7000 \times (1+8\%)^3 = 8817.98$$

$$4000 \times (1+8\%)^2 = 4665.60$$

$$4000 \times (1+8\%)^1 = 4320$$

$$4000 \times (1+8\%)^0 = 4000$$

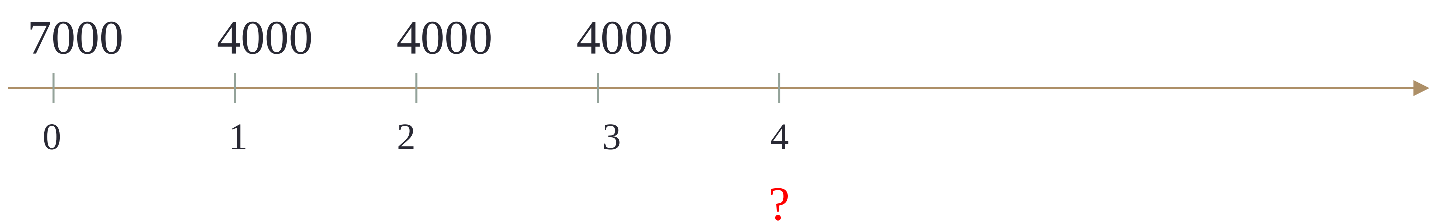
Total value in 3 years = 8,817.98 + 4,665.60 + 4,320.00 + 4,000.00 = 21,803.58

Multiple cash flows – Future value

You think you will be able to deposit 4,000 at the end of each of the next three years in a bank account paying 8% interest

You currently have 7,000 in the account

How much will you have in 4 years?



$7000 \times (1+8\%)^4 = 9523.42$

$4000 \times (1+8\%)^3 = 5038.83$

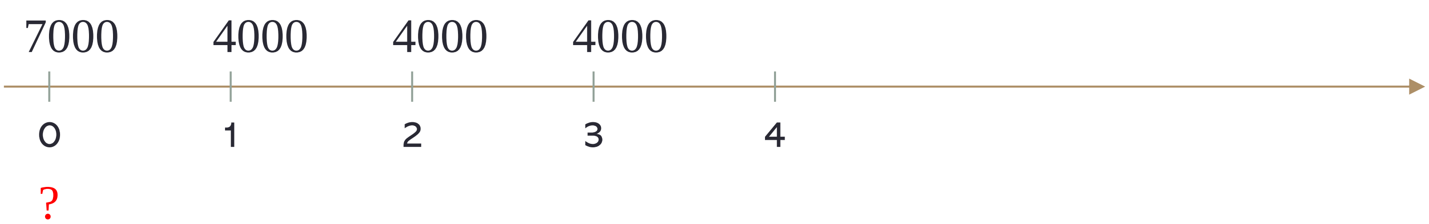
$4000 \times (1+8\%)^2 = 4665.60$

$4000 \times (1+8\%)^1 = 4320$

Total value in 4 years = 9523.42+5038.85 + 4,665.60 + 4,320.00 = 23547.87

Multiple cash flows – Future value

- You think you will be able to deposit 4,000 at the end of each of the next three years in a bank account paying 8% interest
- You currently have 7,000 in the account
- What is the present value of the cash flows?



Present value at year 0= $7000+\frac{4000}{(1+8\%)^1}+\frac{4000}{(1+8\%)^2}+\frac{4000}{(1+8\%)^3}$ =17308.39

Multiple cash flows – Present value

$$PV = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots \text{for N years} = \sum_{n=1}^{N} \frac{C_n}{(1+r)^n}$$

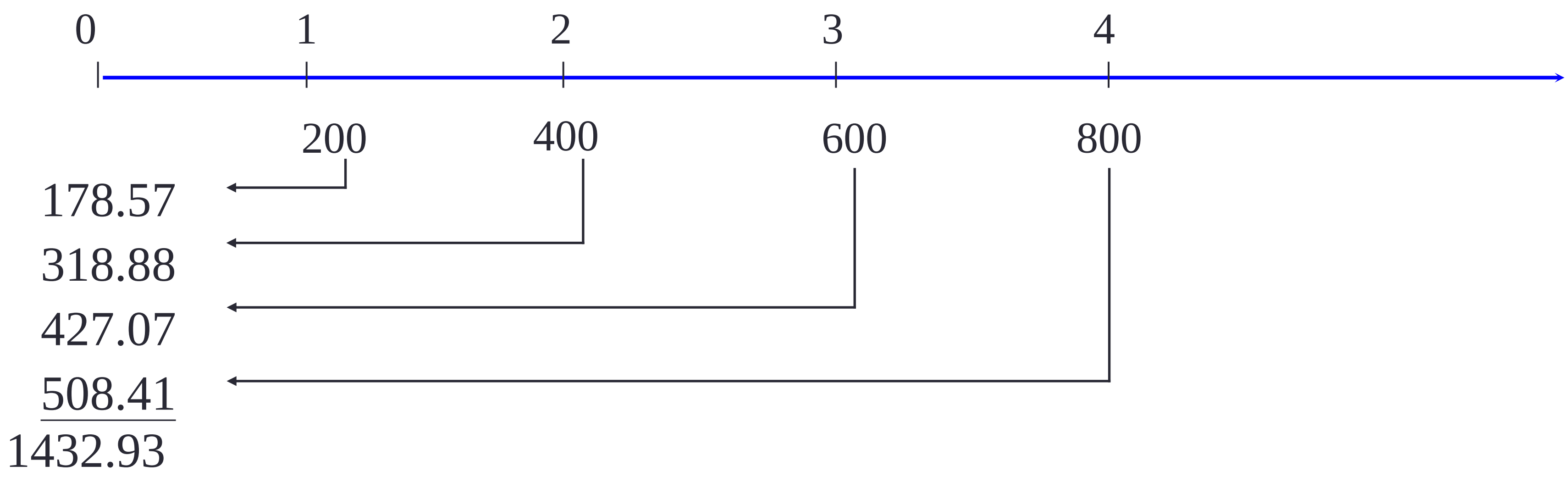
- C_n is a stream of unequal cash flow: C_1 at the end of year 1, C_2 at the end of year 2 and so on.

Multiple cash flows – Present value

Investment opportunity example 1

- You are offered an opportunity to buy a commercial property with the rental income of £200k in one year, £ 400k in two years, £600k in three years, and you can sell it for £800k in four years.
- You can earn 12% on very similar investments.
- What is the most you should pay for the property?

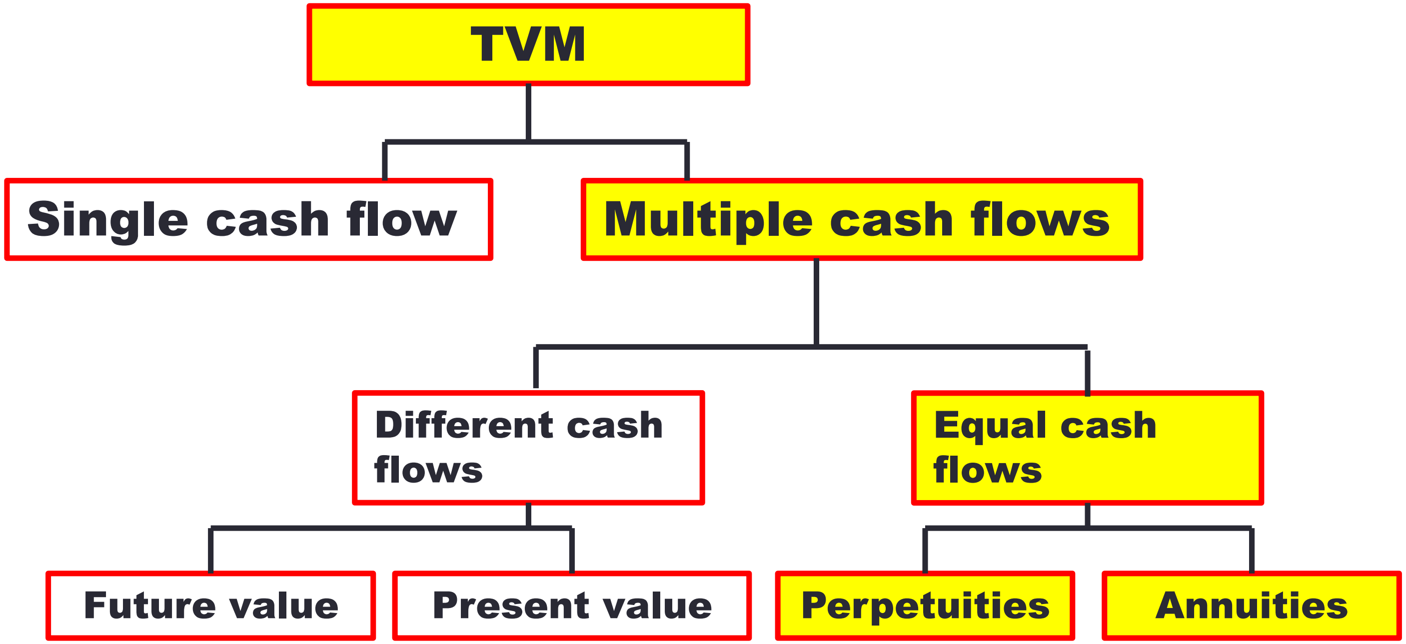
Multiple cash flows – Present value



Find the PV of each cash flow and add them

- Year 1 CF: 200 / $(1.12)^1$ = 178.57
- Year 2 CF: 400 / $(1.12)^2$ = 318.88
- Year 3 CF: 600 / $(1.12)^3$ = 427.07
- Year 4 CF: 800 / $(1.12)^4$ = 508.41
- Total PV = 178.57 + 318.88 + 427.07 + 508.41 = 1432.93

Time value of money



Which options do you prefer?

Rate is 4%

A. You are offered 40,000 today.

B. You are offered 10,000 every year for the next five years.

C. You are offered 2,000 every year forever (indefinitely).

A. 40k
B. 44.52k
=10/1.04+10/1.04^2+10/1.04^3+10/1.04^4+10/1.04^5
C. 2/1.04+2/1.04^2+…………………………………..=50k

Indeed, we have a formula to figure out the value of cash flow streams if the cash flows follow a certain pattern.

Annuity and Perpetuity

Annuity – finite series of EQUAL payments that occur at regular intervals

Perpetuity – infinite series of equal payments

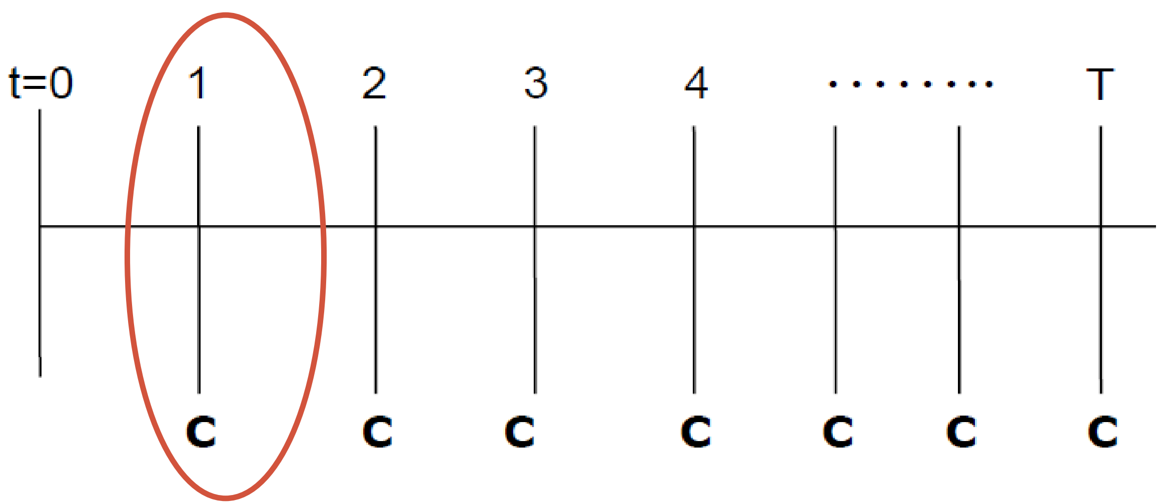
Annuities

Applicable when

1. the values of cash flow are the same in all periods
2. the discount rates are the same in all periods
3. **<u>Note the PV formula for annuities assume that first cash flow occurs next year (year 1)</u>**

$$PV = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T}$$

PV (t =0)



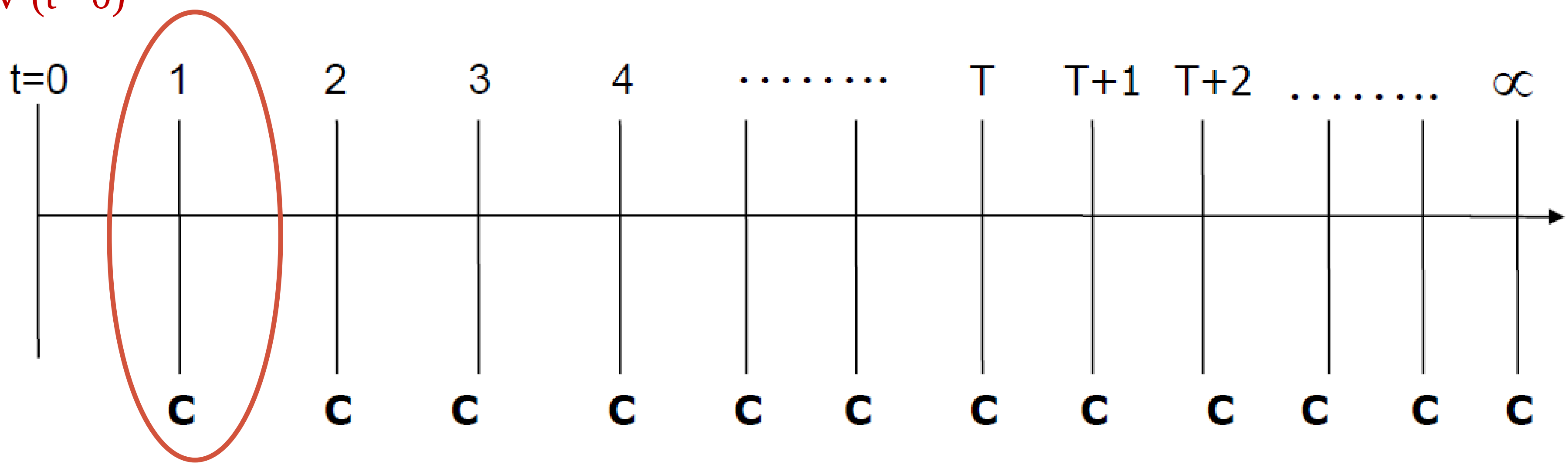
Perpetuities

Applicable when

1. the values of cash flow are the same in all periods, and last forever
2. the discount rates are the same in all periods
3. **Note the PV formula for perpetuities assume that first cash flow occurs next year (year 1)**

$$PV = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^\infty}$$

PV (t =0)



Annuities and Perpetuities Formula (1/2)

$$PV = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T}$$

We have a formula for the above:

Annuities $(\boldsymbol{T = a\ finite\ number})$**:**

$$\boldsymbol{PV}\ \textbf{of annuity} = C \times \left(\frac{1 - \frac{1}{(1+r)^T}}{r} \right) = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T} \right)$$

$$PV = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^\infty}$$

We have a formula for the above:

Perpetuity $(\boldsymbol{T} \rightarrow \infty)$**:**

$$\boldsymbol{PV}\ \textbf{of perpetuity} = \frac{C}{r}$$

Annuities and Perpetuities Formula (2/2)

- **Perpetuities** $(\boldsymbol{T \rightarrow \infty})$**:**

$$PV \text{ of perpetuity} = \frac{C}{r}$$

- **Annuities** $(\boldsymbol{T = a\ finite\ \ number})$**:**

$$PV \text{ of annuity} = C \times \left(\frac{1 - \frac{1}{(1+r)^T}}{r} \right) = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T} \right)$$

$$FV \text{ of annuity} = \frac{C}{r} \times \left((1+r)^T - 1 \right)$$

$$\times (1+r)^T$$

Which options do you prefer?

Rate is 4%, you face the following offers.

A. You are offered 40k today.

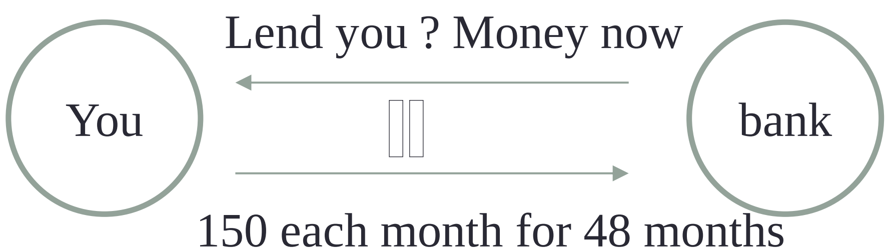
B. You are offered 10k every year for the next five years.

C. You are offered 2k every year forever (indefinitely).

- What are the present value of those offers? Use the formula for perpetuity and annuity.

Annuities – Present Value (1/2)

You can afford to pay 150 per month towards a car. The bank can lend you the money at 1% per month for 48 months. How much can you borrow?



You are borrowing money today, so you need to compute the present value

$$PV = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T}\right) = \frac{150}{0.01} \times \left(1 - \frac{1}{(1+0.01)^{48}}\right)$$

$$= 5{,}696.09$$

5696.09 now = 150 each month for next 48 months

Annuities – Present Value (2/2)

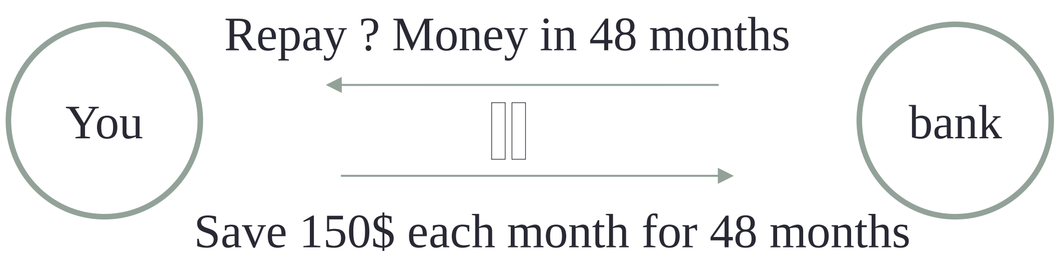
Suppose you win the *Euromillions* jackpot of 10 million

- You have two options:
- 1) receive 10 million today
- 2) The money is paid in equal annual instalments of 350,000 over 30 years (350,000 x 30=10.5 million)

If the appropriate discount rate is 5%, which option do you choose? (think about it for 5 minutes)

Annuities – Future Value (1/2)

You can afford to pay 150 per month into a saving account of a bank. The bank pays interests at 1% per month for 48 months. How much can you withdraw from the bank in 48 months?



You are withdrawing money in the future, so you need to compute the future value

$$FV = \frac{C}{r} \times \left((1+r)^T - 1\right) = \frac{150}{0.01} \times \left((1+0.01)^{48} - 1\right)$$
$$= 9183.39$$

9183.39 in 48 months = 150 each month for next 48 months

Recall that we compute the present value (= 5,696.09), what is the relationship?

Answer: 9183.39 = 5696.09 x $(1+0.01)^{48}$

Annuities – Future Value (2/2)

- Suppose you begin saving for your retirement by depositing \$2,000 per year in a savings account
- If the interest rate is 7.5%, how much can you withdraw in 40 years? (think about it for 5 minutes)

Note that

$$FV \text{ of annuity} = \frac{C}{r} \times \left((1+r)^T - 1\right)$$

Finding the Annuity Payment -> C

- Suppose you want to borrow $10,000 for a new car
- You can borrow at 8% per year, compounded monthly (8/12 =0.6667% per month)
- If you take a 4-year loan, what is your monthly payment?

Try to solve it.

Note that $$PV \text{ of annuity} = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T}\right)$$

Annuity – Finding Number of Payments

- Suppose you borrow 2,000 at 5% and you are going to make annual payments of 734.42. How long before you pay off the loan? (don't look at the answer, try to solve it by yourself).
- Note that PV of annuity $= \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^{T}}\right)$

Annuity – Finding the Interest Rate

- Suppose you borrow 10,000 from your parents to buy a car. You agree to pay 207.58 per month for 60 months. What is the monthly interest rate?

$$10{,}000 = 207.58 \times \frac{1 - \frac{1}{(1+r)^{60}}}{r}$$

- Solve for r

$$r = 0.75\%$$

- This is not simple manually
- In Excel: =RATE(60,207.58,-10000)

A Case of House Mortgage – Finding the Annuity (C)

I want to purchase a house in Hamburg with the price of 300,000 euros. The bank offers me a plan with an interest rate of 1%. What is my annual payment (annuity) for with no down payment for a 30-year loan?

Exercise Q5 – Q7

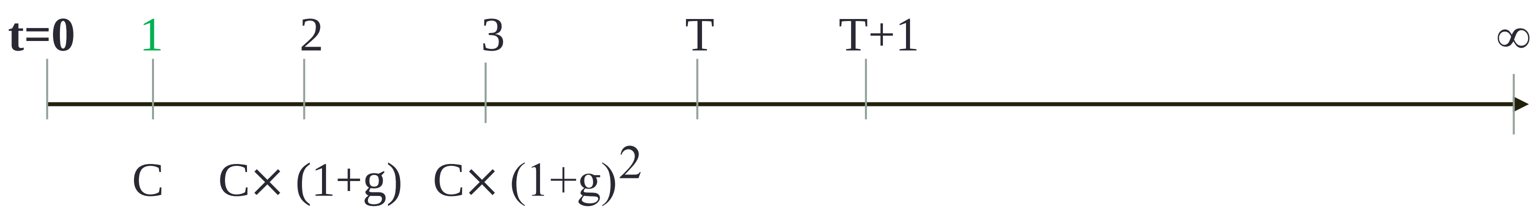
Q5. I am buying a stock which promised me dividend payments of 100$ every year. What is the (present) value of the stock (how much do I pay for the stock) with a rate of return of 10%?

Q6. What is your monthly payment (compounded monthly) if you take a 10-year loan for your house which costs 100,000$ with an annual rate of 12%.

Q7. Suppose you borrow a student loan of $7,000 at 3% annual rate and you are going to make a monthly payment of $90 (monthly compounded). How long does it take before you can pay off the loan?

Growing Annuities and Perpetuities

What if the payment is not constant, but grows at a steady rate of *g*…



Assume you receive a cash flow of 100 at the end of year **1**, and it grows at 5%

100   $100 \times (1+5\%) = 105$ $100 \times (1+5\%)^2 = 110.25$

$$PV(t=0) = \frac{C}{(1+r)^1} + \frac{C \times (1+g)}{(1+r)^2} + \cdots + \frac{C \times (1+g)^{T-1}}{(1+r)^T}$$

Growing Annuities and Perpetuities (1/3)

$$PV = \frac{C}{(1+r)^1} + \frac{C \times (1+g)}{(1+r)^2} + \cdots + \frac{C \times (1+g)^{T-1}}{(1+r)^T}$$

We have a formula for the above:

Growing Annuities ($T = a\, finite\, number$):

$$PV \text{ of growing annuity} = \frac{C}{r-g} \times \left(1 - \left(\frac{1+g}{1+r}\right)^T\right)$$

Growing Perpetuity ($T \to \infty$):

$$PV \text{ of growing perpetuity} = \frac{C}{r-g}$$

Growing Annuities and Perpetuities (2/3)

- Present Value of a Growing Perpetuity

$$PV = \frac{C}{r-g}$$

Example of growing perpetuity: increasing dividend payments of an established firm.

- Value of a share: In 2020, You held a share of Starbucks that pay you a dividend 10$ at the end of 2021, and Starbucks decide to increase the dividend by 5% every year (which indicates that the dividend will become 10.5 in 2022, 11.025 in 2023 and so on). Assume the required rate of return for Starbucks is 10%. What is the value (present value) of the Starbucks share (how much are you willing to pay for the share of Starbucks)?

Answer: 10 / (10%-5%)= 200 $

Growing Annuities and Perpetuities (3/3)

- Present Value of a Growing Annuity

$$PV = \frac{C}{r-g} \times \left(1 - \left(\frac{1+g}{1+r}\right)^T\right)$$

Example of growing annuity: increasing pensions/savings contribution by 3% till retirement.

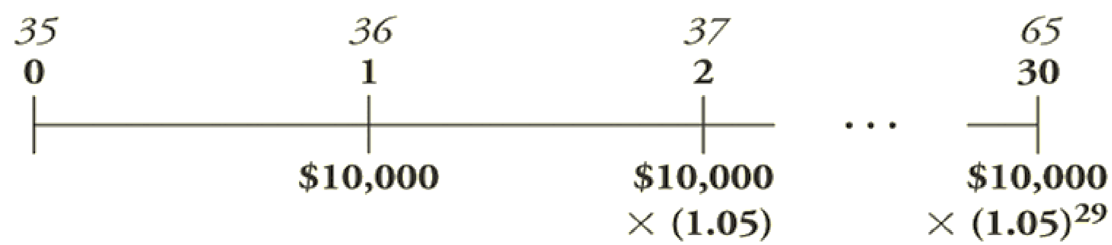
- Pension plan: In 2020, Your employer plan to save 10k next year (in 2021) in your pension pot and increase the pension contributions by 5% every year until the year of 2023 (i.e., 10.5k in 2022, 11.025k in 2023). Assume the required rate of return is 10%. Most pension calculators show numbers that are in "today's money", based on what money is worth & what things cost today.
- What is the value of your pension pot in 2020?

Answer: $\frac{10}{10\%-5\%} \times (1 - \frac{(1+5\%)^3}{(1+10\%)^3})$= 26.05k , or another way: $\frac{10}{(1+10\%)} + \frac{10.5}{(1+10\%)^2} + \frac{11.025}{(1+10\%)^3}$ = 26.05k

Growing Annuity – Example

Ellen, 35, plans to contribute to her retirement saving account yielding 10% return. She would start with a $10,000 payment at the age of 36, but plans to increase the contribution by 5% every year until 65. What is the value (present value) of her retirement pot?

Her new savings plan is represented by the following timeline:



Annuities and Perpetuities in the Future

What if the annuity / perpetuity payments occur in the far future rather than next period, what is PV (year 0) of them?

t = 0 1 2 **3** 4 5 6 ∞

$PV_0 = \frac{PV_3}{(1+r)^3}$ $PV_3 = \frac{C}{r-g}$ C C(1+g) C(1+g)^2.................

Assume you receive a cash flow of 100 starting from the end of year **4 (rather than year 1)**, and it grows at 5% indefinitely

100 100×(1+5%)=105 100× (1+5%)^2=110.25

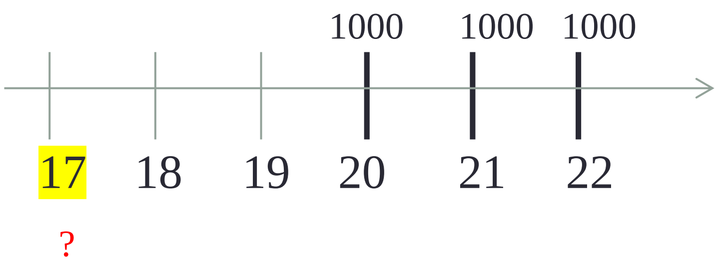
Step one: Apply annuity formla $PV(year\ 3) = \frac{C}{r-g}$ to obtain the present value at the end of year 3

Step two: Convert them into present value at year 0: $PV_0 = \frac{\boldsymbol{PV(year\ 3)}}{\boldsymbol{(1+r)^3}}$

Annuities in the Future – Example

I am currently 17 years old, my parents set up a college fund that pays me 1000$ every year from the age of 20 until I am 22.

How much do my parents need to invest (what is the (present/current) value of the college fund) when I am 17 years old with the rate of 10%.

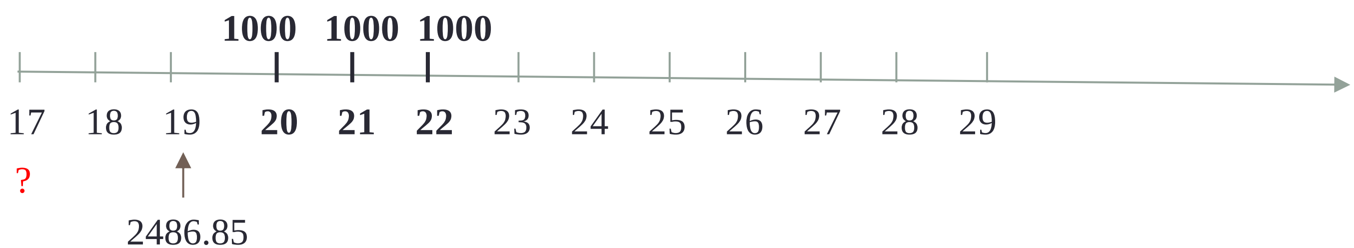


The present value of the cash flow streams when I am 17 years old:

$$\frac{1000}{(1+10\%)^3}+\frac{1000}{(1+10\%)^4}+\frac{1000}{(1+10\%)^5}=2055.25$$

Annuities in the Future – Example

I am currently 17 years old, my aunt set up a college fund that pays me 1000$ every year from the age of 20 until I am 22. What is the (present/current) value of the fund with the interest rate of 10%.



1) Determine what is T? How many payments do I receive, 2 or 3?

$22 - 20 + 1 = 3$ number of cash flows

2) PV(at the age of 19) $= \frac{1000}{10\%}(1 - \frac{1}{1.1^3}) = 2486.85$

3) 2486.85 is the value of payments when I am 19 (note that PV formula of annuity gives us the value *one* period backward).

PV(at the age of 17) $= \frac{2486.85}{(1+10\%)^2} = 2055.25$

$$PV \text{ of annuity} = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T}\right)$$

Annuities in the Future – Example

I am currently 17 years old, my aunt set up a college fund that pays me 1000$ every year from the age of 20 until I am 22. What is the (present/current) value of the fund with the interest rate of 10%.

$$PV_{\underline{\text{(at the age of 19)}}} = \frac{1000}{10\%}\left(1 - \frac{1}{1.1^3}\right) = 2486.85$$

$$PV_{\underline{\text{(at the age of 17)}}} = \frac{2486.85}{(1+10\%)^2} = 2055.25$$

$$\frac{1000}{10\%}\left(1 - \frac{1}{1.1^3}\right) \Rightarrow \frac{1000}{(1+10\%)^1} + \frac{1000}{(1+10\%)^2} + \frac{1000}{(1+10\%)^3} = 2486.85$$

$$\frac{\frac{1000}{10\%}\left(1 - \frac{1}{1.1^3}\right)}{(1+10\%)^2} \Rightarrow \frac{1000}{(1+10\%)^3} + \frac{1000}{(1+10\%)^4} + \frac{1000}{(1+10\%)^5} = 2055.25$$

Exercise Q8

Q8. I am buying a stock in the year of 2020. The stock promised me payment of 1$ every year since the year of 2023. What is the present value of the share (how much will I pay for the share) as of the year 2020 if the rate of return is 10%? (Example of perpetuity)

Some Exercise

Q9. I am buying a share in the year of 2020. The share promised me payment of 1$ in the year of 2023 and grows at a rate of 5% thereafter. What is the present value of the share as of the year 2020 if the rate of return is 10%?

Q10. I am currently 17 years old, my aunt will give me 10,000$ as a gift starting from the age of 20 and the gifts will grow at a rate of 5% until I am 39. What is the value of the gift with the discount rate of 10%.

Solving exercises in Excel

- Number of periods: NPER
- Payment per period: PMT
- Rate : RATE
- Present value: PV
- Future value: FV

- Google Excel Sheet “Time value of money”

Example

I want to purchase a house in Hamburg with the price of 300,000 euros. The bank offers me a plan with an annual interest rate of 6%, payable every month (monthly compounded). What is my monthly payment for with no down payment for a 30-year loan. What is Excel Spreadsheet Formula for solving this financial problem?

	NPER	RATE	PV(cash inflow, borrowing)	FV	PMT (cash outflow, monthly payment)
Given	30 x 12 =360	6%/12 = 0.5%	300,000	0	**− PMT(0.5%, 360,300000)**
Solve					1798.65

Excel Exercise

Recall that
Invest=save (money outflow)
Withdraw=have money (money inflow)

Q1. Your firm plans to buy a warehouse for 100,000$. The bank offers you a 30-year loan with equal annual payments and an interest rate of 8% per year. The bank requires that your firm pay 20% of the purchase price as a down payment, so you can borrow the rest. What is the annual loan payment?

7106.19

Q2. Jessica has just graduated with her MBA. She decided to go into business for herself. A bank was so impressed with Jessica that it has decided to fund her business. In return for an initial business loan of $1 million, Jessica has agreed to pay back 125,000 at the end of each year for the next 30 years, what is the internal rate of return?

12.09%

Q3. You are saving a down payment for a house. You have 10,500 saved already, and you can afford to save an additional 5000 per year at the end of each year. If you earn 7.25% per year on your savings, how long will it take for you to have 60,000?

6.92

Q4. Suppose that you invest 20,000 in an account paying 8% interest. You plan to withdraw 2000 at the end of each year for 15 years. How much money will it be left in the account after 15 years?

9139.15

Q5. You want to purchase a new car and you are willing to pay 10,000. If you can invest at 10% per year and you currently have 7500, how long will it be before you have enough money to pay cash for the car?

Recommend you to write down the Excel formula.
E.g, – PMT(0.08,30,80000,0)

Formula Sheet of Excel:
Payment per period: PMT (rate,nper,pv,[fv],[type])
Present value: PV (rate, nper, pmt,[fv],[type])
Future value: FV (rate, nper, [pmt], [pv], [type])
Number of payment: NPER (rate, pmt, pv, [fv], [type])
Net Present Value: NPV (rate, value1, value 2,…)
Internal rate of return: IRR(value 1, value 2,.)

Rate(nper,pmt,pv,[fv],[type])

Past - Exam Question

- You are saving a down payment for a house. You have now already saved 10,000 and plan to save 1,000 per year at the end of each year. If you earn 5% per year on your savings, how long will it take to have 100,000. Which is the excel formula to solve the above problem?
  - A. NPER(0.05, −1000, −10000,100000)
  - B. NPER(0.05,1000, −10000,100000)
  - C. NPER(0.05,1000,10000,100000)
  - D. None of the above

Formula Sheet of Excel:
Payment per period: PMT (rate,nper,pv,[fv],[type])
Present value: PV (rate, nper, pmt,[fv],[type])
Future value: FV (rate, nper, [pmt], [pv], [type])
Number of payment: NPER (rate, pmt, pv, [fv], [type])
Net Present Value: NPV (rate, value1, value 2,…)
Internal rate of return: IRR(value 1, value 2,.)

Present value for a regular pattern of cash flow streams – Summary

Conditions	**Formula**
Infinite periods / perpetuity *Constant*/stable cash flows	$PV = \frac{C}{r}$
Infinite periods / perpetuity *Growing* cash flows	$PV = \frac{C}{r-g}$
Finite periods Constant/stable cash flows	$PV = \frac{C}{r} \times \left(1 - \frac{1}{(1+r)^T}\right)$
Finite periods *Growing* cash flows	$PV = \frac{C}{r-g} \times \left(1 - \left(\frac{1+g}{1+r}\right)^T\right)$

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